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Orthostructures from Sesquilinear Forms. A Primer

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We make a linkage with the "ultimate" generalization of Hilbert space, an R-module with an orthogonality relation, and certain constructs due to Foulis and Randall and other related structures suggested by Cattaneo, Franco, and Marino.

1. INTRODUCTION

The primordial mathematical model of the logic of a quantum mechanical system is the lattice of closed subspaces of a complex infinite-dimensional Hilbert space. The inner product (sesquilinear form) is instrumental in providing a sense of negation (orthocomplementation $M \mapsto M^{\perp}$) by providing a relation of orthogonality between subspaces. The central role of the orthomodular identity is now well recognized. From this seminal model, many generalizations have evolved. One notable line of research has been developed by Foulis and Randall (1972, 1973). In the present paper, we make a linkage with the "ultimate" generalization of Hilbert space, an R-module with an orthogonality relation, and certain constructs due to Foulis and Randall and other related structures suggested by Cattaneo *et al.* (1987).

2. LINEAR ORTHOGONALITY RELATIONS ON MODULES

Let R be a ring with unity 1 and let ${}_RM$ be a left R-module. Let $\mathrm{Lat}({}_RM)$ denote the complete modular lattice of submodules of M ordered by set inclusion. A relation $\perp$ on M is called a *linear orthogonality relation* on M

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provided

$$\text{if } x \perp y, \text{ then } y \perp x \text{ for all } x, y \text{ in } M \tag{2.1}$$

$$\text{if } x \perp y \text{ and } x \perp z, \text{ then } x \perp (y+z) \tag{2.2}$$

$$\text{if } x \perp y, \text{ then } x \perp ay \text{ for all } a \text{ in } R \tag{2.3}$$

$$x \perp y \text{ for all } y \text{ in } M \text{ iff } x = \vec{0} \tag{2.4}$$

This last condition is one of "nondegeneracy." If S is any subset of M, define the *orthogonal of* S by $S^{\perp} = \{x \in M \mid x \perp s \text{ for all } s \in S\}$. Clearly, a nondegenerate symmetric relation $\perp$ on the module M is a linear orthogonality relation if and only if $\{x\}^{\perp}$ is a submodule of M for all x in M. Call the pair $({}_RM, \perp)$ an *orthomodule*. The orthogonality relation distinguishes two kinds of elements in M. If $x \in M$ and $x \perp x$, then x is called *isotropic*; otherwise x is called *anisotropic*. Call $({}_RM, \perp)$ *anisotropic* provided M admits no nonzero isotropic elements.

The orthogonality relation also distinguishes various families of submodules. We present a taxonomy. Let $({}_RM, \perp)$ be an orthomodule. The *semisimple submodules* are given by

$$L_{ss}({}_RM, \perp) = \{F \in \mathrm{Lat}({}_RM) \mid F \cap F^{\perp} = (\vec{0})\}$$

When we restrict the orthogonality relation to a semisimple submodule, we obtain another orthomodule. Now, $L_{ss}({}_RM, \perp)$ is partially ordered by inclusion and is a bounded poset [$(\vec{0})$ and M belong] with an orthogonality relation. Moreover, any orthogonal family of semisimple submodules has a join in $L_{ss}({}_RM, \perp)$. This poset is "algebraic" in the sense that any upward-directed family of semisimple submodules has a join in $L_{ss}({}_RM, \perp)$.

In the Hilbert space example, we are interested in the closed subspaces. This leads us to consider the *orthoclosed submodules*

$$L_c({}_RM, \perp) = \{F \in \mathrm{Lat}({}_RM) \mid F = F^{\perp\perp}\}$$

Here we have a complete lattice with an involution; the meet is $\bigsqcap F_\alpha = \bigcap F_\alpha$ and the join is $\bigsqcup F_\alpha = (\sum F_\alpha)^{\perp\perp} = (\bigcup F_\alpha)^{\perp\perp}$, where the F_α are all closed. The map $F \mapsto F^{\perp}$ is not necessarily an orthocomplementation on $L_c({}_RM, \perp)$. We could restrict our attention to the closed and semisimple submodules, but the orthomodularity we seek actually resides elsewhere.

Let $({}_RM, \perp)$ be an orthomodule. A submodule F of M is called *splitting* if $F + F^{\perp} = M$. Let

$$L_S({}_RM, \perp) = \{F \in \mathrm{Lat}({}_RM) \mid F + F^{\perp} = M\}$$

Clearly $(\vec{0})$ and M belong to $L_S({}_RM, \perp)$ so we have a bounded poset. Note that $L_S({}_RM, \perp) \subseteq L_{ss}({}_RM, \perp)$ and $F^\perp$ is splitting whenever F is. A useful characterization of splitting submodules is given by the following result.

Theorem 2.1. Let $({}_RM, \perp)$ be an orthomodule. Then F is a splitting submodule of M if and only if for all submodules N of M with $F \subseteq N$ we have $N = F + (N \cap F^\perp)$.

Proof. Suppose first that F is splitting. Let N be any submodule of M containing F. Then $F \subseteq N$, so by the modular law, $(F \vee G) \wedge N = F \vee (G \wedge N)$ for any submodule G. Choose $G = F^\perp$. Then $(F + F^\perp) \cap N = F + (F^\perp \cap N)$. That is,

$$N = M \cap N = (F + F^\perp) \cap N = F + (F^\perp \cap N)$$

Conversely, suppose the condition. Take $M = N \supseteq F$. Then $M = F + (M \cap F^\perp) = F + F^\perp$. The proof is complete. ■

It follows easily now that if F is splitting and $F \subseteq N$ with $N \cap F^\perp = (\vec{0})$ then $F = N$; that is, F is maximal in the set of all submodules N with $N \cap F^\perp = (\vec{0})$. It also follows that

$$L_S({}_RM, \perp) \subseteq L_c({}_RM, \perp) \cap L_{ss}({}_RM, \perp)$$

Theorem 2.2. $L_S({}_RM, \perp)$ is an orthomodular poset.

Proof. First suppose F and G are splitting with $F \perp G$. Then $F \subseteq G^\perp$, so $G^\perp = F + (G^\perp \cap F^\perp)$. Thus

$$M = G + G^\perp = G + (F + (G^\perp \cap F^\perp)) = (G + F) + (G + F)^\perp$$

That is, $G + F$ is splitting and must be the least upper bound of F and G in $L_S({}_RM, \perp)$. Therefore this poset is orthogonally disjunctive.

Next, the map $F \mapsto F^\perp$ on $L_S({}_RM, \perp)$ has the properties $F = F^{\perp\perp}$, $F \subseteq G$ implies $G^\perp \subseteq F^\perp$, $F \cap F^\perp = (\vec{0})$, and $F + F^\perp = F \vee F^\perp = M$. Finally, for the orthomodular identity, let $F \subseteq G$ in $L_S({}_RM, \perp)$. Then $G = F + (G \cap F^\perp) = F + (G^\perp + F)^\perp = F \vee (G^\perp \vee F)^\perp$ and we are done. ■

We pause in our development to offer some examples. The most natural way to generate a linear orthogonality relation is by using some kind of "inner product."

Example 2.3. Take any classical inner product space, real, complex, or quaternionic, complete or not, $(V, \langle\cdot,\cdot\rangle)$. Define $x \perp y$ when $\langle x, y\rangle = 0$ for $x, y \in V$. This yields a linear orthogonality relation on V (Gudder, 1974; Gudder and Holland, 1975).

More generally, take R a ring with involution * and let Φ be a nondegenerate orthosymmetric $[\Phi(x, y) = 0$ implies $\Phi(y, x) = 0]$*-sesquilinear form on an R-module M. Define $x \perp y$ iff $\Phi(x, y) = 0$. This produces a linear orthogonality relation on M. Call (M, Φ) a *quadratic module* (Piziak, 1973).

Example 2.4. For a concrete example, let $R = \mathbb{Z}$, the ring of integers, and let $M = \{(a_1, a_2, a_3, \ldots) \mid a_i \in \mathbb{Z}, a_i = 0$, except for finitely many $i\}$. Then M is a $\mathbb{Z}$-module under slotwise operations. Define $\Phi((a_i), (b_i)) = \sum_{i=1}^{\infty} a_i b_i$. Then Φ is a nondegenerate symmetric bilinear form on M which admits no isotropic elements. Thus (M, Φ) is a quadratic $\mathbb{Z}$-module. In particular, $L_C(M, \Phi)$ is a nonmodular orthocomplemented lattice. Note that this module is free, since $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, \ldots), \ldots$ is a basis. Consider another sequence $f_0 = e_1, \ldots, f_k = e_1 + e_2 + \cdots + e_k - ke_{k+1}$. One computes that

$$\Phi(f_0, f_j) = 1 \qquad \text{for all} \quad j \geq 0$$

$$\Phi(f_i, f_j) = 0 \qquad \text{for all} \quad i, j \geq 1, \quad i \neq j$$

$$\Phi(f_j, f_j) = j + j^2 \qquad \text{for all} \quad j \geq 1$$

and that $\{f_0, f_1, f_2, \ldots\}$ is another basis for this module. Let

$$F = \text{span}\{f_1, f_3, f_5, \ldots\}$$

$$G = \text{span}\{f_2, f_4, f_6, \ldots\}$$

$$H = \text{span}\{f_1, f_2, f_3, \ldots\}$$

Then $F = G^{\perp}$ and $G = F^{\perp}$, so F and G are closed but not splitting. Moreover, $H^{\perp} = (\bar{0})$, $H^{\perp\perp} = E$, and $L_S(M, \Phi)$ is an orthomodular poset which is not a lattice.

3. ORTHOGONAL SETS IN ANISOTROPIC ORTHOMODULES

In this section, for simplicity, let $({}_R M, \perp)$ denote an anisotropic orthomodule. Let S be an arbitrary submodule of M. Various kinds of orthogonal sets can be distinguished and will be used in the next section to generate ordered sets.

1. *The orthogonal subsets of S*, $\mathcal{O}(S, \perp)$:

$$\mathcal{O}(S, \perp) = \{A \subseteq S \backslash (\bar{0}) \mid x, y \in A, x \neq y \text{ implies } x \perp y\}$$

This set is partially ordered by inclusion and has the property that any subset of an element of $\mathcal{O}(S, \perp)$ is again an element of $\mathcal{O}(S, \perp)$. The union of any upward-directed family of elements in $\mathcal{O}(S, \perp)$ is in $\mathcal{O}(S, \perp)$, so $\mathcal{O}(S, \perp)$ is inductive and hence has maximal elements. We shall take singletons to be orthogonal sets.

2. *The maximal orthogonal subsets of S, $\mathcal{MO}(S, \perp)$*:

$$\mathcal{MO}(S, \perp) = \{A \in \mathcal{O}(S, \perp) \mid A \subseteq B,\ B \in \mathcal{O}(S, \perp) \text{ implies } A = B\}$$

3. *The total orthogonal subsets of S, $\mathcal{TO}(S, \perp)$*:

$$\mathcal{TO}(S, \perp) = \{A \in \mathcal{O}(S, \perp) \mid x \in S,\ x \perp A \text{ implies } x = \vec{0}\}$$

4. *The basic orthogonal subsets of S, $\mathcal{BO}(S, \perp)$*:

$$\mathcal{BO}(S, \perp) = \{A \in \mathcal{O}(S, \perp) \mid A^{\perp\perp} = S\}$$

5. *The regular orthogonal subsets of S, $\mathcal{RO}(S, \perp)$*:

$$\mathcal{RO}(S, \perp) = \{A \in \mathcal{O}(S, \perp) \mid A = B \cup C,\ B \cap C = \varnothing \text{ implies } B^{\perp\perp} = C^{\perp}\}$$

6. *The Dacey orthogonal subsets of S, $\mathcal{DO}(S, \perp)$*:

$$\mathcal{DO}(S, \perp) = \{A \in \mathcal{O}(S, \perp) \mid A \subseteq x^{\perp} \cup y^{\perp},\ x, y \in M \text{ implies } x \perp y\}$$

The following facts are easily established.

Lemma 3.1. Let $({}_R M, \perp)$ be an anisotropic orthomodule and let $S \subseteq M$. Then:

(i) If $A \in \mathcal{MO}(S, \perp)$, then $A^{\perp} \cap S \subseteq (\vec{0})$.
(ii) If F is a submodule of M and $A \in \mathcal{MO}(F, \perp)$, then $A^{\perp} \cap F = (\vec{0})$.
(iii) $\mathcal{MO}(M, \perp) \subseteq \mathcal{BO}(M, \perp)$.

Theorem 3.2. Let $({}_R M, \perp)$ be an anisotropic orthomodule and let $S \subseteq M$. Then

$$\mathcal{DO}(S, \perp) = \mathcal{RO}(S, \perp)$$

Proof. Let $A \in \mathcal{DO}(S, \perp)$. We shall show $A \in \mathcal{RO}(S, \perp)$. Suppose $A = B \cup C$ with $B \cap C = \varnothing$. We show $B^{\perp\perp} = C^{\perp}$. Now A is an orthogonal set, so $C \subseteq B^{\perp}$ and so $B^{\perp\perp} \subseteq C^{\perp}$ or equivalently $C^{\perp\perp} \subseteq B^{\perp}$. It suffices to show $B^{\perp} \subseteq C^{\perp\perp}$. Take $x \in B^{\perp}$. Show that $x \perp y$ for all y in $C^{\perp}$. Take any y in $C^{\perp}$. Then $B^{\perp\perp} \subseteq x^{\perp}$ and $C^{\perp\perp} \subseteq y^{\perp}$ and $A = B \cup C \subseteq B^{\perp\perp} \cup C^{\perp\perp} \subseteq x^{\perp} \cup y^{\perp}$. But A is Dacey, so $x \perp y$.

Conversely, suppose A is regular and suppose $A \subseteq x^{\perp} \cup y^{\perp}$. Take $B = A \cap x^{\perp}$ and $C = A \backslash B$. Then $A = B \cup C$ and $B \cap C = \varnothing$. Since A is regular, $C^{\perp} = B^{\perp\perp}$. Now clearly $B \subseteq x^{\perp}$. We claim $C \subseteq y^{\perp}$. Let $z \in C$. Then $z \in A \backslash B$, so $z \in A$ and $z \notin B$. Then $z \in x^{\perp} \cup y^{\perp}$, so $z \in x^{\perp}$ or $z \in y^{\perp}$. But if $z \in x^{\perp}$, then $z \in A \cap x^{\perp} = B$, a contradiction, so $z \in y^{\perp}$. Thus $x \in B^{\perp}$ and $y \in C^{\perp} = B^{\perp\perp}$, hence $x \perp y$. ■

Lemma 3.3. Use the assumptions above. If $A \in \mathscr{R}\mathscr{O}(S, \perp)$ and $B \in \mathscr{O}(S, \perp)$ with $B \supseteq A$, then $B^{\perp} = (\vec{0})$.

Proof. Now $A \subseteq B$ means $B = A \cup (B \setminus A)$. But any superset of a Dacey set is Dacey. Thus B is Dacey and hence regular. Being regular, $A^{\perp\perp} = (B \setminus A)^{\perp}$. Thus

$$B^{\perp} = (A \cup (B \setminus A))^{\perp} = A^{\perp} \cap (B \setminus A)^{\perp} = A^{\perp} \cap A^{\perp\perp} = (\vec{0})$$

by anisotropy. ■

The next theorem summarizes the relationships between the various kinds of orthogonal subsets of an anisotropic orthomodule. Since the way has been prepared, the details of the proof will be omitted.

Theorem 3.4. For an anisotropic orthomodule $({}_R M, \perp)$,

$$\mathscr{D}\mathscr{O}(M, \perp) = \mathscr{R}\mathscr{O}(M, \perp) \subseteq \mathscr{B}\mathscr{O}(M, \perp) = \mathscr{T}\mathscr{O}(M, \perp) = \mathscr{M}\mathscr{O}(M, \perp)$$

4. FOULIS–RANDALL SUBMODULES OF AN ANISOTROPIC ORTHOMODULE

In this section, let $({}_R M, \perp)$ denote an anisotropic orthomodule. There is a natural mapping $\psi: \mathscr{O}(M, \perp) \to L_c(M, \perp)$ given by $\psi(A) = A^{\perp\perp}$. Clearly, ψ preserves order and $\psi(A) = M$ for all $A \in \mathscr{M}\mathscr{O}(M, \perp)$. The image of ψ defines the collection of Foulis–Randall submodules of M. So a submodule F of M is a *Foulis–Randall submodule* iff there is an $A \in \mathscr{O}(M, \perp)$ such that $A^{\perp\perp} = F$. Let

$$\mathrm{FR}(M, \perp) = \{F \in \mathrm{Lat}(M) \mid F = A^{\perp\perp} \text{ for some } A \in \mathscr{O}(M, \perp)\}$$

Clearly, $M \in \mathrm{FR}(M, \perp)$ and we agree to accept $(\vec{0}) \in \mathrm{FR}(M, \perp)$ also.

Theorem 4.1. $\mathrm{FR}(M, \perp)$ is completely orthogonally disjunctive. That is, let $\{F_\alpha\}$ be a family of pairwise orthogonal submodules in $\mathrm{FR}(M, \perp)$. Then $\sup\{F_\alpha\}$ exists in $\mathrm{FR}(M, \perp)$ and moreover $\sup\{F_\alpha\} = \bigsqcup F_\alpha$ as computed in $L_c(M, \perp)$.

Proof. Let $\{F_\alpha\}$ be a family of pairwise orthogonal elements in $\mathrm{FR}(M, \perp)$. Then for each α, $F_\alpha = A_\alpha^{\perp\perp}$ for some $A_\alpha \in \mathscr{O}(M, \perp)$. Let $A = \bigcup A_\alpha$; we claim $A \in \mathscr{O}(M, \perp)$. Let $x, y \in A$, $x \neq y$. Then there exist α and β with $x \in A_\alpha$, $y \in A_\beta$. If $\alpha = \beta$, x and y are in the same orthogonal set, so $x \perp y$. If $\alpha \neq \beta$, then $F_\alpha \perp F_\beta$, so $F_\alpha \subseteq F_\beta^{\perp}$. That is, $A_\alpha^{\perp\perp} \subseteq A_\beta^{\perp}$, so $A_\alpha \subseteq A_\beta^{\perp}$, whence $x \perp y$. Now let $G = A^{\perp\perp}$. Then $G \in \mathrm{FR}(M, \perp)$. Also, $A_\alpha \subseteq A$ for each α, so $A_\alpha^{\perp\perp} \subseteq A^{\perp\perp}$. Thus $F_\alpha \subseteq G$ for all α, and so G is an upper bound for $\{F_\alpha\}$ in $\mathrm{FR}(M, \perp)$. Is G the least upper bound? Let $H \in \mathrm{FR}(M, \perp)$ with $H \supseteq F_\alpha$ for all α. Then $H \supseteq F_\alpha \supseteq A_\alpha$ for all α, so $H \supseteq \bigcup A_\alpha = A$. Thus $H = H^{\perp\perp} \supseteq A^{\perp\perp} = G$, so G is the least upper bound of the $\{F_\alpha\}$ in $\mathrm{FR}(M, \perp)$. Finally,

$$\bigsqcup F_\alpha = (\sum F_\alpha)^{\perp\perp} = \langle \bigcup F_\alpha \rangle^{\perp\perp} = (\bigcup F_\alpha)^{\perp\perp} = (\bigcap F_\alpha^\perp)^\perp$$
$$= (\bigcap A_\alpha^\perp)^\perp = (\bigcup A_\alpha)^{\perp\perp} = A^{\perp\perp} = G \quad \blacksquare$$

To establish the next theorem we need the following result.

Lemma 4.2. Let $F \in \mathrm{FR}(M, \perp)$. Suppose $F = A^{\perp\perp}$ for some $A \in \mathcal{O}(M, \perp)$. If $B \in \mathcal{MO}(F^\perp, \perp)$, then $A \cup B \in \mathcal{MO}(M, \perp)$.

Proof. Since $B \subseteq F^\perp = A^\perp$, we have $A \cap B = \varnothing$ and $A \cup B \in \mathcal{O}(M, \perp)$. Then $(A \cup B)^{\perp\perp} = (A^\perp \cap B^\perp)^\perp = (F^\perp \cap B^\perp)^\perp$. Let $\vec{0} \neq x \in F^\perp \cap B^\perp$. Then $x \in F^\perp$ and $x \in B^\perp$, so $B \cup \{x\}$ is in $\mathcal{O}(F^\perp, \perp)$. By maximality, $x \in B$, so $x \in B^\perp \cap B$, so $x = \vec{0}$, a contradiction. Thus $F^\perp \cap B^\perp = (\vec{0})$, so $(A \cup B)^{\perp\perp} = (\vec{0})^\perp = M$, hence $A \cup B$ is an $\mathcal{MO}(M, \perp)$, as was to be shown. $\blacksquare$

Recall that a *local complement* of F in $\mathrm{FR}(M, \perp)$ is any G in $\mathrm{FR}(M, \perp)$ with $F \perp G$ and $\sup\{F, G\} = M$. With the help of the lemma above, the next theorem is straightforward, so the proof will be omitted.

Theorem 4.3. Let $F, G \in \mathrm{FR}(M, \perp)$ with $F = A^{\perp\perp}$, $G = B^{\perp\perp}$. Then F is a local complement of G iff $A \cup B \in \mathcal{MO}(M, \perp)$.

We culminate our development in this section with the following result.

Theorem 4.4. $\mathrm{FR}(M, \perp)$ is an orthologic; that is, the following hold:

(i) If $\{F_1, F_2, F_3\}$ is a pairwise orthogonal set, then $F_1 \perp \sup\{F_2, F_3\}$.
(ii) Every $F \in \mathrm{FR}(M, \perp)$ has at least one local complement in $\mathrm{FR}(M, \perp)$.

The proof follows familiar lines laid down by Foulis and Randall.

In general, it does not appear to be possible to prove that if $F \in \mathrm{FR}(M, \perp)$, then $F^\perp \in \mathrm{FR}(M, \perp)$. The best this author can offer at this point is the following.

Lemma 4.5. Let $F \in \mathrm{FR}(M, \perp)$ with $F = A^{\perp\perp}$. Suppose $F^\perp$ is in $\mathrm{FR}(M, \perp)$. Then there exists $B \supseteq A$, $B \in \mathcal{MO}(M, \perp)$, such that

$$F^\perp = (B \backslash A)^{\perp\perp}$$

There are other "Foulis–Randall-like" submodules that are of interest, but space does not allow us to pursue them here. The interested reader may wish to study the collection of *mighty Foulis–Randall submodules*:

$$\mathrm{MFR}(M, \perp) = \{F \in \mathrm{Lat}(M) \mid F = A^{\perp\perp} \text{ for all } A \in \mathcal{MO}(F, \perp)\}$$

or the collection of *super-Foulis–Randall submodules*

$$\mathrm{SFR}(M, \perp) = \{F \in \mathrm{Lat}(M) \mid F = A^{\perp\perp} \text{ for all } A \in \mathcal{MO}(F, \perp) \text{ and } F^{\perp} = B^{\perp\perp} \text{ for any } B \in \mathcal{MO}(F^{\perp}, \perp)\}$$

Clearly, we have a nesting

$$\mathrm{SFR}(M, \perp) \subseteq \mathrm{MFR}(M, \perp) \subseteq \mathrm{FR}(M, \perp) \subseteq L_c(M, \perp)$$

5. DACEY MODULES

As before, let $({}_RM, \perp)$ be an anisotropic orthomodule. Call this orthomodule *Dacey* iff every maximal orthogonal set is Dacey.

Lemma 5.1. Let $(M, \perp)$ be Dacey and $F \in \mathrm{FR}(M, \perp)$ with $F = A^{\perp\perp}$ for some $A \in \mathcal{O}(M, \perp)$. Let $B \supseteq A$ with $B \in \mathcal{MO}(M, \perp)$. Then $F^{\perp}$ is in $\mathrm{FR}(M, \perp)$ and $F^{\perp} = (B \backslash A)^{\perp\perp}$.

Proof. Suppose $F = A^{\perp\perp}$ for some $A \in \mathcal{O}(M, \perp)$. Use Zorn's Lemma to extend A to B in $\mathcal{MO}(M, \perp)$. Then $B = A \cup (B \backslash A)$ and $A \cap (B \backslash A) = \varnothing$ and B is Dacey, whence regular. Thus $A^{\perp\perp} = (B \backslash A)^{\perp}$. That is, $F = A^{\perp\perp} = (B \backslash A)^{\perp}$, so $F^{\perp} = (B \backslash A)^{\perp\perp}$. ■

Theorem 5.2. Let $(M, \perp)$ be an anisotropic orthomodule. Then the following statements are all equivalent:

(i) $(M, \perp)$ is Dacey.
(ii) Every F in $\mathrm{FR}(M, \perp)$ has a unique local complement which is in fact $F^{\perp}$.
(iii) $\mathrm{FR}(M, \perp)$ is an orthomodular poset under $F \mapsto F^{\perp}$.

Proof. (5.2.1) ⇒ (5.2.2). Suppose $(M, \perp)$ is Dacey. If $F \in \mathrm{FR}(M, \perp)$, then $F^{\perp} \in \mathrm{FR}(M, \perp)$ by the above lemma, and $F^{\perp}$ is a local complement since $F \subseteq F^{\perp\perp}$, so $F \perp F^{\perp}$ and $F \sqcup F^{\perp} = (F + F^{\perp})^{\perp\perp} = (F^{\perp} \cap F^{\perp\perp})^{\perp} = (\bar{0})^{\perp} = M$. Now suppose G is some local complement of F. Then $G \perp F$ and $G \sqcup F = M$. Thus $G \subseteq F^{\perp}$. Now $G = A^{\perp\perp}$ and $F = B^{\perp\perp}$ for orthogonal sets A and B. Note $A \subseteq A^{\perp\perp} = G \subseteq F^{\perp} = B^{\perp}$, so $A \cup B$ is an orthogonal set and $A \cap B = \varnothing$. Also, $M = G \sqcup F = (A \cup B)^{\perp\perp}$, so $A \cup B$ is maximal orthogonal, hence Dacey, hence regular. Thus $A^{\perp\perp} = B^{\perp}$, i.e., $G = F^{\perp}$.

(5.2.2) ⇒ (5.2.3). Left to the reader.

(5.2.3) ⇒ (5.2.1). Suppose $\mathrm{FR}(M, \perp)$ is an orthomodular poset under $F \mapsto F^{\perp}$. Let $A \in \mathcal{MO}(M, \perp)$. We shall show A is regular and hence Dacey. Let $A = B \cup C$ with $B \cap C = \varnothing$. Let $F = B^{\perp\perp}$ and $G = C^{\perp\perp}$. Now $B \subseteq C^{\perp}$, since A is an orthogonal set, so $B^{\perp\perp} \subseteq C^{\perp}$, so $F \subseteq C^{\perp} = G^{\perp}$. By orthomodularity, $G^{\perp} = F \sqcup (G^{\perp} \cap F^{\perp})$, so

$$G = G^{\perp\perp} = (F \sqcup (G^{\perp} \cap F^{\perp}))^{\perp} = F^{\perp} \cap (G^{\perp} \cap F^{\perp})^{\perp}$$
$$= F^{\perp} \cap (G \sqcup F)^{\perp\perp} = F^{\perp} \cap (G \sqcup F)$$

Now $M = A^{\perp\perp} = (B \cup C)^{\perp\perp} \subseteq (F + G)^{\perp\perp} = F \sqcup G$, hence $M = F \sqcup G$, and so $G = F^{\perp} \cap M = F^{\perp}$, so $G^{\perp} = F^{\perp\perp} = F$, i.e., $C^{\perp} = B^{\perp\perp}$. Therefore A is regular. ■

Finally, we note that the Dacey condition is tied into the stronger types of Foulis–Randall submodules.

Theorem 5.3. Let $(M, \perp)$ be an anisotropic orthomodule. Then $(M, \perp)$ is Dacey if and only if $\mathrm{MFR}(M, \perp) = \mathrm{FR}(M, \perp)$.

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REFERENCES

Cattaneo, G., Franco, G., and Marino, G. (1987). Ordering on families of subspaces of pre-Hilbert spaces and Dacey pre-Hilbert spaces, *Bollettino U.M.I.* (7), **1-B**, 167–183.

Foulis, D. J., and Randall, C. H. (1972). Operational statistics I: Basic concepts, *Journal of Mathematical Physics*, **13**, 1667.

Foulis, D. J. and Randall, C. H. (1973). Operational statistics II: Manuals of operations and their logic, *Journal of Mathematical Physics*, **14**, 1472.

Gudder, S. P. (1974). Inner product spaces, *American Mathematical Monthly*, **81**, 29–36; "correction," *American Mathematical Monthly*, **82**, 251–252 (1975).

Gudder, S. P., and Holland, Jr., S. (1975). Second correction to "Inner product spaces," *American Mathematical Monthly*, **82**, 818.

Piziak, R. (1973). Orthomodular posets from sesquilinear forms, *Journal of the Australian Mathematical Society*, **15**, 265–269.